SEPARATION OF TWO-PARTICLE EQUATION INTO CENTER-OF-MASS AND RELATIVE MOTION

$$-\frac{\hbar^2}{2m_1}\frac{\partial^2\psi'(x_1,x_2)}{\partial x_1^2}-\frac{\hbar^2}{2m_2}\frac{\partial^2\psi'(x_1,x_2)}{\partial x_2^2}+U(x_2-x_1-r_e)\psi'(x_1,x_2)=E\psi'(x_1,x_2)$$

define $\quad x_r = x_2 - x_1 - r_e \quad\quad \mu = \dfrac{m_1 m_2}{m_1+m_2}$

$$x_{cm} = \frac{m_1}{M}x_1 + \frac{m_2}{M}x_2 \qquad M = m_1 + m_2$$

$$\psi'(x_1,\ x_2) = \psi(x_r,\ x_{cm})$$

$$\left(\frac{\partial\psi'}{\partial x_1}\right)_{x_2} = \left(\frac{\partial\psi}{\partial x_r}\right)_{x_{cm}}\left(\frac{\partial x_r}{\partial x_1}\right)_{x_2} + \left(\frac{\partial\psi}{\partial x_{cm}}\right)_{x_r}\left(\frac{\partial x_{cm}}{\partial x_1}\right)_{x_2}$$

$$= (-1)\left(\frac{\partial\psi}{\partial x_r}\right) + \frac{m_1}{M}\left(\frac{\partial\psi}{\partial x_{cm}}\right)$$

$$= -\left(\frac{\partial\psi}{\partial x_r}\right) + \frac{m_1}{M}\frac{\partial\psi}{\partial x_{cm}}$$

$$\frac{\partial^2\psi'}{\partial x_1^2} = \frac{\partial}{\partial x_r}\left[-\frac{\partial\psi}{\partial x_r} + \frac{m_1}{M}\frac{\partial\psi}{\partial x_{cm}}\right]\frac{\partial x_r}{\partial x_1}$$

$$+\frac{\partial}{\partial x_{cm}}\left[-\frac{\partial\psi}{\partial x_r} + \frac{m_1}{M}\frac{\partial\psi}{\partial x_{cm}}\right]\frac{\partial x_{cm}}{\partial x_1}$$

$$\frac{\partial^2\psi'}{\partial x_1^2} = \left[-\frac{\partial^2\psi}{\partial x_r^2} + \frac{m_1}{M}\frac{\partial^2\psi}{\partial x_r \partial x_{cm}}\right](-1)$$

$$+\left[-\frac{\partial^2\psi}{\partial x_{cm}\partial x_r} + \frac{m_1}{M}\frac{\partial^2\psi}{\partial x_{cm}^2}\right]\left[\frac{m_1}{M}\right]$$

$$\boxed{\frac{\partial^2\psi'}{\partial x_1^2} = \frac{\partial^2\psi}{\partial x_r^2} + \left(\frac{m_1}{M}\right)^2\frac{\partial^2\psi}{\partial x_{cm}^2} - 2\frac{m_1}{M}\frac{\partial^2\psi}{\partial x_{cm}\partial x_r}}$$

Now for $\dfrac{\partial^2\psi'}{\partial x_2^2}$

$$\frac{\partial\psi'}{\partial x_2} = \left(\frac{\partial\psi}{\partial x_r}\right)_{x_{cm}}\left(\frac{\partial x_r}{\partial x_2}\right)_{x_1} + \left(\frac{\partial\psi}{\partial x_{cm}}\right)_{x_r}\left(\frac{\partial x_{cm}}{\partial x_2}\right)_{x_1}$$

$$= \left(\frac{\partial\psi}{\partial x_r}\right) + \frac{m_2}{M}\frac{\partial\psi}{\partial x_{cm}}$$

$$\frac{\partial^2 \psi'}{\partial x_2^2} = \left[\frac{\partial^2 \psi}{\partial x_r^2} + \frac{m_2}{M}\frac{\partial^2 \psi}{\partial x_r \partial x_{cm}}\right]$$

$$+ \left[\frac{\partial^2 \psi}{\partial x_{cm} \partial x_r} + \frac{m_2}{M}\frac{\partial^2 \psi}{\partial x_{cm}^2}\right]\left[\frac{m_2}{M}\right]$$

$$\boxed{\frac{\partial^2 \psi'}{\partial x_2^2} = \frac{\partial^2 \psi}{\partial x_r^2} + \left(\frac{m_2}{M}\right)^2 \frac{\partial^2 \psi}{\partial x_{cm}^2} + \frac{2m_2}{M}\frac{\partial^2 \psi}{\partial x_{cm}\partial x_r}}$$

Now taking $-\frac{\hbar^2}{2m_1}\frac{\partial^2 \psi'}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2 \psi'}{\partial x_2^2} + U(x_r)\psi'$

$$-\frac{\hbar^2}{2}\left[\frac{1}{m_1}\frac{\partial^2 \psi}{\partial x_r^2} + \frac{m_1}{M^2}\frac{\partial^2 \psi}{\partial x_{cm}^2} - 2\left(\frac{1}{M}\right)\frac{\partial^2 \psi}{\partial x_{cm}\partial x_r}\right]$$

$$-\frac{\hbar^2}{2}\left[\frac{1}{m_2}\frac{\partial^2 \psi}{\partial x_r^2} + \frac{m_2}{M^2}\frac{\partial^2 \psi}{\partial x_{cm}^2} + 2\left(\frac{1}{M}\right)\frac{\partial^2 \psi}{\partial x_{cm}\partial x_r}\right]$$

$$+U(x_r)\psi = E\psi$$

$$\text{using } \frac{m_1 + m_2}{M^2} = \frac{1}{M}$$

$$\frac{-\hbar^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{\partial^2 \psi}{\partial x_r^2} - \frac{\hbar^2}{2}\left(\frac{1}{M}\right)\frac{\partial^2 \psi}{\partial x_{cm}^2} + U(x_r)\psi = E\psi$$

$$\frac{-\hbar^2}{2\mu}\frac{\partial^2 \psi}{\partial x_r^2} - \frac{\hbar^2}{2M}\frac{\partial^2 \psi}{\partial x_{cm}^2} + U(x_r)\psi = E\psi$$

Now one can use separation of variables

$$\psi(x_r, x_{cm}) = \psi_r(x_r)\psi_{cm}(x_{cm})$$

$$-\frac{\hbar^2}{2\mu}\frac{1}{\psi_r}\frac{\partial^2 \psi_r}{\partial x_r^2} - \frac{\hbar^2}{2M}\frac{1}{\psi_{cm}}\frac{\partial^2 \psi_{cm}}{\partial x_{cm}^2} + U(x_r) = E$$

$$\underbrace{-\frac{\hbar^2}{2\mu}\frac{1}{\psi_r}\frac{\partial^2 \psi_r}{\partial x_r^2} + U(x_r)}_{E_r} = \underbrace{E + \frac{\hbar^2}{2M}\frac{1}{\psi_{cm}}\frac{\partial^2 \psi_{cm}}{\partial x_{cm}^2}}_{E_r}$$

$$\boxed{-\frac{\hbar^2}{2\mu}\frac{\partial^2\psi_r}{\partial x_r^2}+U(x_r)\psi_r(x_r)=E_r\psi_r(x_r)}$$

RELATIVE MOTION OF PARTICLE MASS μ

$$-\frac{\hbar^2}{2M}\frac{\partial^2\psi_{cm}}{\partial x_{cm}^2}=\underbrace{(E-E_r)}\psi_{cm}(x_{cm})$$

call this $E_{cm} = E - E_r$

$E = E_{cm} + E_r$

$$\boxed{-\frac{\hbar^2}{2M}\frac{\partial^2\psi_{cm}}{\partial x_{cm}^2}=E_{cm}\psi_{cm}(x_{cm})}$$

CENTER-OF-MASS TRANSLATIONAL MOTION OF PARTICLE MASS M