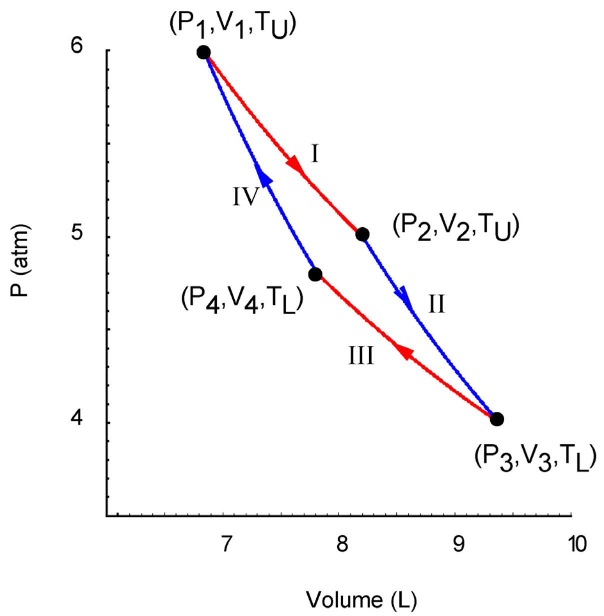
Carnot Arithmetic
(*ideal gas, Carnot Cycle*)

HW#2 Problem 10



P_1= 6 atm, T_1=T_U=500K

P_2= 5 atm, T_2=T_U= 500K

q_{II}=0

P_3=4 atm, T_3=T_L

P_4=4.8 atm, T_4=T_L

q_{IV}=0

STEP I Isothermal expansion, T_U, $V_1 \rightarrow V_2$:

$$\Delta U_I = 0 \tag{1.1}$$

$$w_I = -nRT_U \ln\frac{V_2}{V_1} = nRT_U \ln\frac{P_2}{P_1} \tag{1.2}$$

$$q_I = -w_I = nRT_U \ln\frac{V_2}{V_1} = nRT_U \ln\frac{P_1}{P_2} \tag{1.3}$$

STEP II Adiabatic Expansion: (T_U,V_2) → (T_L, V_3)

$$V_3 = V_2\left(\frac{T_U}{T_L}\right)^{\frac{\overline{C_V}}{R}} \qquad V_3 = V_2\left(\frac{P_2}{P_3}\right)^{\frac{\overline{C_V}}{\overline{C_P}}} \tag{2.1}$$

$$T_L = T_U\left(\frac{V_2}{V_3}\right)^{\frac{R}{\overline{C_V}}} \qquad T_L = T_U\left(\frac{P_3}{P_2}\right)^{\frac{R}{\overline{C_P}}} \tag{2.2}$$

$$P_3 = P_2\left(\frac{V_3}{V_2}\right)^{\frac{\overline{C_P}}{\overline{C_V}}} \qquad P_3 = P_2\left(\frac{T_L}{T_U}\right)^{\frac{\overline{C_P}}{R}} \tag{2.3}$$

$$q_{II} = 0; \qquad w_{II} = \Delta U \tag{2.4}$$

$$\Delta U_{II} = n\overline{C}_V \Delta T = n\overline{C}_V (T_L - T_U)$$

$$\Delta U_{II} = n\overline{C}_V T_U \left(\left(\frac{V_2}{V_3} \right)^{\frac{R}{\overline{C}_V}} - 1 \right) = n\overline{C}_V T_U \left(\left(\frac{P_3}{P_2} \right)^{\frac{R}{\overline{C}_P}} - 1 \right) \tag{2.5}$$

STEP III Isothermal Compression: (T_L, V_3) → (T_L, V_4)

$$\Delta U_{III} = 0 \tag{3.1}$$

$$w_{III} = -nRT_L \ln \frac{V_4}{V_3} = nRT_L \ln \frac{P_4}{P_3} \tag{3.2}$$

$$q_{III} = -w_{III} = nRT_L \ln \frac{V_4}{V_3} = nRT_L \ln \frac{P_3}{P_4} \tag{3.3}$$

STEP IV Adiabatic compression: (T_L, V_4) → (T_U, V_1)

$$V_4 = V_1 \left(\frac{T_U}{T_L} \right)^{\frac{\overline{C}_V}{R}} \qquad V_4 = V_1 \left(\frac{P_1}{P_4} \right)^{\frac{\overline{C}_V}{\overline{C}_P}} \tag{4.1}$$

$$T_L = T_U \left(\frac{V_1}{V_4} \right)^{\frac{R}{\overline{C}_V}} \qquad T_L = T_U \left(\frac{P_4}{P_1} \right)^{\frac{R}{\overline{C}_P}} \tag{4.2}$$

$$P_4 = P_1 \left(\frac{V_1}{V_4} \right)^{\frac{\overline{C}_P}{\overline{C}_V}} \qquad P_4 = P_1 \left(\frac{T_L}{T_U} \right)^{\frac{\overline{C}_P}{R}} \tag{4.3}$$

$$q_{IV} = 0; \qquad w_{IV} = \Delta U_{IV} \tag{4.4}$$

$$\Delta U_{IV} = n\overline{C}_V \Delta T = n\overline{C}_V (T_U - T_L)$$

$$\Delta U_{IV} = n\overline{C}_V T_U \left(1 - \left(\frac{V_1}{V_4} \right)^{\frac{R}{\overline{C}_V}} \right) = n\overline{C}_V T_U \left(1 - \left(\frac{P_4}{P_1} \right)^{\frac{R}{\overline{C}_P}} \right) \tag{4.5}$$

TOTAL CYCLE:

$$\Delta U_{total} = 0 \tag{T.1}$$

$$w_{total} = w_I + w_{II} + w_{III} + w_{IV}$$

note that the work done on the adiabatic steps 2 and 4 are equal in magnitude but opposite in sign.

Let us assume that T_U , T_L , P_1, and P_2 are given, and thus the remaining P's and V's are determined, and write the net work and heat in terms of these conditions:

$$w_I = nRT_U \ln\frac{P_2}{P_1} \qquad (\text{from} \quad 1.2) \qquad (T.2)$$

$$w_{III} = nRT_L \ln\frac{P_4}{P_3} \qquad (\text{from} \quad 3.2)$$

$$P_4 = P_1\left(\frac{T_L}{T_U}\right)^{\frac{\overline{C_P}}{R}} \qquad (\text{from} \quad 4.3)$$

$$P_3 = P_2\left(\frac{T_L}{T_U}\right)^{\frac{\overline{C_P}}{R}} \qquad (\text{from} \quad 2.3)$$

$$\frac{P_4}{P_3} = \frac{P_1}{P_2} \qquad (T.3)$$

thus

$$w_{III} = nRT_L \ln\frac{P_1}{P_2}$$

and

$$w_{total} = w_I + w_{III}$$

$$w_{total} = nRT_U \ln\frac{P_2}{P_1} + nRT_L \ln\frac{P_1}{P_2} = nRT_U \ln\frac{P_3}{P_4} + nRT_L \ln\frac{P_4}{P_3} \qquad (T.4)$$

$$w_{total} = nR\left(T_U - T_L\right)\ln\frac{P_2}{P_1} = nR\left(T_U - T_L\right)\ln\frac{P_3}{P_4}$$

$$q_{total} = -w_{total} \qquad (T.5)$$

EFFICIENCY:

since $P_1 < P_0$ (expansion) $w_{total} < 0$; i.e. work done ***by system on surroundings***
Efficiency is defined as the ratio of heat input (step 1) to total work done on surroundings. The heat exchange in step 3 is “wasted heat”, lost to surroundings at T_L as thermal pollution.

$$\varepsilon = \frac{-w_{total}}{q_{input}} \qquad (E.1)$$

$$\varepsilon = \left(\frac{-nR\left(T_U - T_L\right)\ln\frac{P_2}{P_1}}{nRT_U \ln\frac{P_1}{P_2}}\right) \qquad (\text{from T.4 and 1.3}) \qquad (E.2)$$

$$\varepsilon = \frac{\left(T_U - T_L\right)}{T_U} = 1 - \frac{T_L}{T_U} \qquad (E.3)$$