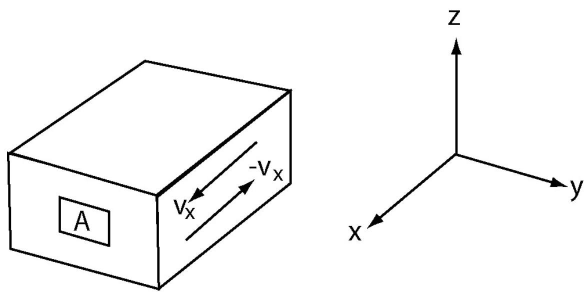
Heuristic Derivation of the Relationship between Temperature and Energy
via the Ideal Gas Law

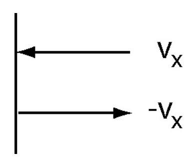
I. Ideal gas law by experiment (Boyle's Law and Charles's Law)

PV=nRT	n=moles, R= gas constant
PV=n*kT	n*=molecules, k=Boltzmann's constant
	$k=\frac{R}{N}$, N=Avogadro's number

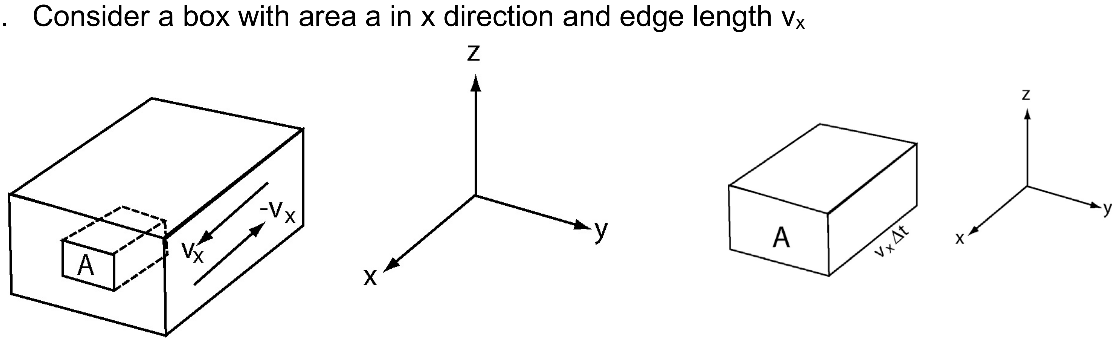
II. Heuristic, but accurate, 'derivation' (box of volume V with n* total molecules)



1. consider 1-D: molecules all with same v_x (all same v_x is 'heuristic')
2. elastic collision with wall velocity of mass goes $v_x \rightarrow -v_x$
3. from physics $P=\frac{F}{A}$ P=pressure F=force A=area
4. from physics $F=\frac{dp}{dt}$ p=mv, momentum t=time
5. $dp \approx \Delta p = m\Delta v = m\ 2v_x$ per collision (m is mass of particle)



6. total Δp in given time Δt, $\left(\frac{\Delta p}{\Delta t}\right) \approx \left(\frac{dp}{dt}\right)$, would depend on number of collisions in that interval.
7. Consider a box with area a in x direction and edge length v_x



½ of all molecules in a rectangular box would collide with A in time Δt (the other ½ are going in a direction away from A)

8. $\frac{n^*}{V}$ is density of molecules, $Av_x\Delta t$ is volume of rectangular box,

$\frac{1}{2}\frac{n^*}{V}Av_x\Delta t$ is number of molecules colliding with area A

9. total $\Delta p=(2mv_x)\left(\frac{1}{2}\right)\left(\frac{n^*}{V}\right)(Av_x\Delta t) = (mv_x{}^2)\left(\frac{n^*}{V}\right)(A\Delta t)$

10. finally

$$P=\frac{F}{A}=\frac{\left(\frac{dp}{dt}\right)}{A}\approx\frac{\left(\frac{\Delta p}{\Delta t}\right)}{A}$$

$$P=\frac{\frac{(mv_x^2)\left(\frac{n^*}{V}\right)(A\Delta t)}{\Delta t}}{A}$$

$$P=(mv_x^2)\left(\frac{n^*}{V}\right)$$

11. Equating P from mechanics with the empirically observed $P=\frac{n^*}{V}kT$

$$P=(mv_x^2)\left(\frac{n^*}{V}\right)=\left(\frac{n^*}{V}\right)kT$$

$$(mv_x^2)=kT$$

this would also be true for area A at the y and z ends of the volume.

12. In 3D the kinetic energy of a particle is $KE=\left(\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}mv_z^2\right)$

and with (heuristically) $|v_x|=|v_y|=|v_z|$

one gets

$$KE=\frac{3}{2}n^*kT=\frac{3}{2}nRT$$

$$KE=\frac{3}{2}kT \text{ per molecule or } \frac{3}{2}RT \text{ per mole}$$

III. The important take home messages are

1. Good warm up of physics and equation derivation
2. For molecule with only kinetic energy (e.g. monatomic species), and ideal gas (no intermolecular forces):

   $$E=\frac{3}{2}kT \text{ per molecule or } E=\frac{3}{2}RT \text{ per mole}$$

3. For monatomic ideal gas, E is function of only T;
   T constant ⇔ E constant